The proof of Mather's theorem goes via a chain of equivalent notions of stability. Especially useful is the following

6. Def

For $f \in C^\infty(M,N)$ denote by $[f]_x$ the germ of f at x.

f is locally infinitesimally stable at x if for every germ of a vector field along f (= germ of a section $M \to f^*TN$), $[v]_x$, there exist germs of vector fields $[s]_x$ in $C^\infty(TM)_x$ and $[t]_{f(x)}$ in $C^\infty(TN)_{f(x)}$, s.t.

$$[v]_x = [df \cdot s]_x + [t \circ f]_x .$$

In local coordinates $(x_1, \dots, x_m)$ around x and $(y_1, \dots, y_n)$ around $f(x)$ this amounts to solving

$$v_i = \sum_{j=1}^{m} \frac{\partial f_i}{\partial x_j} s_j + t_i(f_1, \dots, f_n) \qquad i = 1, \dots, n$$

7. Thm: f is loc. inf. stable at x if these equations can be solved to order $n = \dim N$,

i.e.

$$v_i = \sum_{j=1}^{m} \frac{\partial f_i}{\partial x_j} s_j + t_i(f_1, \dots, f_n) + O(\|(x_1, \dots, x_m)\|^{n+1}).$$

Proof: Generalized Malgrange Preparation Theorem (see G&G) $\square$

Note that

1. loc. inf. stability at x is determined by $j^{n+1}f(x)$!

2. this reduces the problem to finite dimensions (can use implicit fctn theorem!)

e.g.: (almost) every singularity we had so far, $x \mapsto x^k$, $(x, y) \mapsto (x, y^2)$, ...

For a vector bundle $\begin{smallmatrix} E \\ \downarrow \pi \\ M \end{smallmatrix}$ consider

$\pi_* : J^k(M,E) \longrightarrow J^k(M,M)$ and $I \subset J^k(M,M)$,

defined by $I := \{ \sigma = [id_M] \}$.

This is a submanifold (and π_* submersion),

so $J^k(E) := \pi_*^{-1}(I)$ is a submfd

of $J^k(M,E)$, the k-jet bundle of

sections of E (a vector bundle over M).

<u>8. Corollary:</u> f loc. inf. stable at x

$$\iff \quad J^n(f^*TN)_x = df(x)\, J^n(TM)_x + f^* J^n(TN)_{f(x)}$$

If f is injective this is equivalent to global (inf.) stability.

In the general case we define for $y \in N$

$$S = \{ x_1, \dots, x_k \} \subset f^{-1}(y)$$

$$J^\ell(E)_S := \bigoplus_{i=1}^{k} J^\ell(E)_{x_i} .$$

f induces maps

- $f^*: J^\ell(TN)_y \longrightarrow J^\ell(f^*TN)_{x_i} \quad \forall i=1,\dots,k$

  and

  $$f^*: J^\ell(TN)_y \longrightarrow J^\ell(f^*TN)_S$$

  $$f^*[t]_y := \left([t\circ f]_{x_1},\dots,[t\circ f]_{x_k}\right)$$

- $df: J^\ell(TM)_{x_i} \longrightarrow J^\ell(f^*TN)_{x_i}$

  and

  $$df: J^\ell(TM)_S \longrightarrow J^\ell(f^*TN)_S$$

  $$df([s_1]_{x_1},\dots,[s_k]_{x_k}) := \left([df\cdot s_1]_{x_1},\dots,[df\cdot s_k]_{x_k}\right)$$

We call f simultaneously locally infinitesimally stable at $x_1,\dots,x_k$ if for all germs of vector fields along f, $[v_1]_{x_1},\dots,[v_k]_{x_k}$, there exist germs of vector fields

$[s_1]_{x_1},\dots,[s_k]_{x_k}$ and $[t]_y$ s.t.

$$[v_i]_{x_i} = [df\, s_i]_{x_i} + [t \circ f]_{x_i} \qquad i=1,\dots,k$$

<u>9. Thm:</u> f is inf. stable if and only if $\forall y \in N$ and every $S \subset f^{-1}(y)$ with $|S| \leq n+1$!

$$J^n(f^*TN)_S = df\left(J^n(TM)_S\right) + f^* J^n(TN)_y$$

<u>Proof:</u> G&G.

Another notion of stability uses homotopies:

<u>10. Def</u> Let $f \in C^\infty(M,N)$, $I_\varepsilon := (-\varepsilon, \varepsilon) \subset \mathbb{R}$

1. A deformation/unfolding of f is a smooth map
$$F : M \times I_\varepsilon \longrightarrow N \times I_\varepsilon$$
$$(x,t) \longmapsto (F_t(x), t)$$
with $F_0 = f$.

2. A deformation/unfolding F of f is trivial if

there is $0 < \delta \leq \varepsilon$ and

$G \in \mathrm{Diff}(M \times I_\delta) \cap \{\text{deformations of } id_M\}$

$H \in \mathrm{Diff}(N \times I_\delta) \cap \{\quad \text{—''—} \quad id_N\}$

s.t.

$$\begin{tikzcd} M \times I_\delta \arrow[r, "F"] \arrow[d, "G"'] \arrow[dr, phantom, "\circlearrowleft"] & N \times I_\delta \arrow[d, "H"] \\ M \times I_\delta \arrow[r, "f \times id_{I_\delta}"'] & N \times I_\delta \end{tikzcd}$$

3. f is stable under deformations/unfoldings /homotopically stable if every deformation of f is trivial.

e.g.: $f(x) = x^2 \qquad F_t(x) = x^2 + t \cdot x$ is trivial

take $G_t(x) = x + \frac{t}{2}$

$H_t(y) = y + \frac{t^2}{4}$

$$\Rightarrow H_t \circ F_t \circ G_t^{-1}(x) = \frac{t^2}{4} + \left(x - \frac{t}{2}\right)^2 + t \cdot \left(x - \frac{t}{2}\right)$$

$$= x^2$$

This defn is motivated by the following idea:

A deformation of f is a map

$c: I_\varepsilon \longrightarrow C^\infty(M,N)$, a curve in

$C^\infty(M,N)$ through f $(c(0)=f)$. Recall

the map $\quad \gamma_f : \mathrm{Diff}(M) \times \mathrm{Diff}(N) \to C^\infty(M,N)$

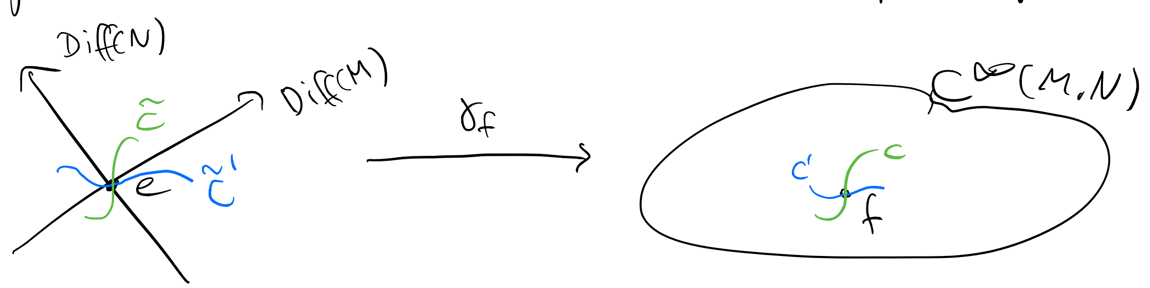
$$(g,h) \mapsto h \circ f \circ g^{-1}.$$

We argued that f is inf. stable iff $d\gamma_f(e)$ is onto.

This can be rephrased by: For every curve $c: t \mapsto F_t$ with $c(0)=f$ there is a curve $\tilde{c}: t \to (G_t, H_t)$ in $\mathrm{Diff}(M) \times \mathrm{Diff}(N)$ with $\tilde{c}(0)=e$ s.t.

$$\gamma_f(G_t, H_t) = H_t \circ f \circ G_t^{-1} = F_t$$

which just means f is homotopically stable!



This can be generalized to k-parameter unfoldings by replacing I_ε with $B_\varepsilon \subset \mathbb{R}^k$.

Q: f not stable (e.g. $x^3, x^4, x^5 \dots$), how "many" non-trivial unfoldings are there?

A: For $\dim M \le 2$ and $k \le 4$ we find Thom's list of seven catastrophes

$$\left(\begin{array}{l} \text{- fold: } x^3 + t x \\ \text{- cusp: } x^4 + t_1 x^2 + t_2 x \\ \text{- swallowtail: } x^5 + t_1 x^3 + t_2 x^2 + t_3 x \\ \quad \dots \end{array} \right)$$

There are even more notions of stability needed for the proof of Mather's theorem, including "transverse stability" which is formulated using $\mathrm{Diff}(M) \times \mathrm{Diff}(N)$ invariant submanifolds of $J^k(M, N)$. For the details, see the books by Arnold or G&G.

Propositions 3, 4 and 5 suggest that stable mappings are always dense in $C^\infty(M,N)$, but in general this is wrong (it is true for C^0-stability).

Mather showed that stable maps are dense in $C^\infty(M,N)$ if and only if for $k := \dim N - \dim M = n-m$ we have

- $n < 7k + 8$ when $k \geq 4$
- $n < 7k + 9$ " $3 \geq k \geq 0$
- $n < 8$ " $k = -1$
- $n < 6$ " $k = -2$
- $n < 7$ " $k \leq -3$